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Orthogonal Multiblade Coordinates

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Introduction

THE method of multiblade coordinates introduced by Coleman[1] is a powerful technique for describing the motion of a multibladed helicopter rotor. Multiblade coordinates describe the overall rotor motion in a nonrotating coordinate frame fixed in the helicopter body. This provides advantages over individual blade coordinates which describe the motion of the individual blades in a frame rotating with the rotor. Some advantages are 1) the overall rotor motion, which includes the motion of all the blades, is intuitively easier to understand because it is described in terms of the collective mode and the periodic longitudinal and lateral modes in the nonrotating frame; 2) fuselage motion and active controls couple with the rotor motion in the form of nonrotating feedback effects; and 3) the transformation to multiblade coordinates replaces some harmonic terms in the equations of motion by constants and higher frequency terms. This last property results in the time-averaged constant coefficient approximation being valid for higher advance ratios compared with the individual blade equations. It also reduces the computation time necessary for a stability analysis using Floquet theory in the case that periodic terms are not time averaged.

The method of multiblade coordinates has been applied by Hohenemser and Yin,[2] Biggers,[3] Gaonkar and Peters,[4,5] Hodges,[6] Briczinski and Cooper,[7] and others to analyze various aspects of the motion of multibladed rotors. In this Note a modified set of multiblade coordinates is introduced for which the transformation between individual blade coordinates and mutiblade coordinates is orthogonal, hence the name orthogonal multiblade coordinates. This property results in simplifications in the transformations of the equations of motion and their solutions.

Multiblade Coordinates

Consider an N-bladed rotor with $N \geq 3$ and let z_k denote the individual blade deflection coordinate (such as flapping angle or lead-lag angle) for the kth blade. Each individual blade coordinate is expressed in terms of the multiblade coordinates as follows[2]:

$$z_k = w_0 + w_d(-1)^k + \sum_{n=1}^{L} w_{nc} \cos n\psi_k + \sum_{n=1}^{L} w_{ns} \sin n\psi_k \quad (1)$$

where $k = 1,2,\ldots,N$ and ψ_k is the azimuth angle of the kth blade. The multiblade coordinate w_0 represents the collective component of the motion, w_d the differential component (which exists only for N even), and w_{nc} and w_{ns} are the cyclic components of various orders. Since only N generalized coordinates are required, the summation upper limit L is equal to $(N-1)/2$ for N odd and $(N-2)/2$ for N even. In terms of a nondimensional time t for which the rotor angular velocity is unity the azimuth angle of each blade is given by

$$\psi_k = t + 2\pi(k-1)/N \quad (2)$$

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Consider the modification of Eq. (1) by including arbitrary multipliers α and β

$$z_k = \alpha w_0 + \alpha w_d(-1)^k + \beta \sum_{n=1}^{L} w_{nc} \cos n\psi_k + \beta \sum_{n=1}^{L} w_{ns} \sin n\psi_k \quad (3)$$

In terms of an N-component vector z of individual blade coordinates

$$z^T = [z_1 z_2 \ldots z_N] \quad (4)$$

and an N-component vector w of multiblade coordinates

$$w^T = [w_0 w_d w_{1c} w_{1s} w_{2c} \ldots w_{Ls}] \quad (5)$$

Equation (3) can be expressed as the linear transformation

$$z(t) = M(t) w(t) \quad (6)$$

By selecting values of the arbitrary multipliers α and β to be

$$\alpha = \sqrt{1/N} \quad (7)$$

$$\beta = \sqrt{2/N} \quad (8)$$

the kth row of the matrix M has the form

$$\frac{1}{\sqrt{N}} [1 (-1)^k \sqrt{2} \cos \Psi_k \sqrt{2} \sin \psi_k \ldots \sqrt{2} \cos L\psi_k \sqrt{2} \sin L\psi_k] \quad (9)$$

where the element $(-1)^k$ is omitted for N odd. Each row of the M matrix, considered as an N vector, then has the property that it has unit magnitude and it is orthogonal to each other row of the matrix. The $N \times N$ transformation matrix M is then an orthogonal matrix, which implies that its inverse is obtained simply by transposing the matrix: $M^{-1} = M^T$. Each row of the matrix M as given in Eq. (9) is then a column of the matrix M^{-1} and a separate closed-form expression for the inverse transformation for the multiblade coordinates in terms of the individual blade coordinates

$$w(t) = M^{-1}(t) z(t) \quad (10)$$

as presented in Ref. 8 is unnecessary. This leads to simplifications in transforming the linear equations of motion into multiblade coordinates since the matrix operations can be expressed solely in terms of the matrix M.

As an example for a three-bladed rotor ($N=3$) the orthogonal matrix M has the form

$$M(t) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \cos\psi_1 & \sqrt{2} \sin\psi_1 \\ 1 & \sqrt{2} \cos\psi_2 & \sqrt{2} \sin\psi_2 \\ 1 & \sqrt{2} \cos\psi_3 & \sqrt{2} \sin\psi_3 \end{bmatrix} \quad (11)$$

It should be emphasized that because of the multipliers α and β the actual collective and differential components of the blade motions are not w_0 and w_d but rather $w_0/\sqrt{N}$ and $w_d/\sqrt{N}$. Similarly the actual cyclic components are $w_{nc}\sqrt{2/N}$ and $w_{ns}\sqrt{2/N}$.

Equations of Motion

To transform the linear equations of motion describing first-order variations in the blade coordinates it is convenient to define a $2N$-component state vector of individual blade coordinates and their time derivatives

$$x^T = [z^T \dot{z}^T] \quad (12)$$

and an analogous state vector in terms of multiblade coordinates

$$y^T = [w^T \dot{w}^T] \tag{13}$$

These state vectors are related by the $2N \times 2N$ matrix $Q(t)$

$$x(t) = Q(t)y(t) \tag{14}$$

where

$$Q(t) = \left[\begin{array}{c|c} M(t) & 0 \\ \hline \dot{M}(t) & M(t) \end{array}\right] \tag{15}$$

A further simplification occurs if one recognizes that the derivative $\dot{M}$ can be expressed in terms of M by

$$\dot{M}(t) = M(t)P \tag{16}$$

where P is a constant skew-symmetric $N \times N$ matrix $(P^T = -P)$. For the case that N is even, the matrix P is block diagonal comprised of $N/2$ skew-symmetric 2×2 partitions along the main diagonal of the form

$$\begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \tag{17}$$

for $n = 0, 1, \ldots, L$. In the case that N is odd there are $(N+1)/2$ partitions of the form of Eq. (17) with the partition for $n = 0$ being a scalar rather than 2×2. As an example for $N = 3$

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{18}$$

Using Eq. (16) the transformation matrix Q becomes

$$Q(t) = \left[\begin{array}{c|c} M(t) & 0 \\ \hline M(t)P & M(t) \end{array}\right] \tag{19}$$

Consider the case of no interblade coupling for which the equation of motion for an individual blade is of the form

$$\ddot{z}_k + c(\psi_k)\dot{z}_k + k(\psi_k)z_k = 0 \tag{20}$$

The equation of motion of the complete system of blades can be expressed in first-order form as

$$\dot{x}(t) = F(t)x(t) \tag{21}$$

where

$$F(t) = \left[\begin{array}{c|c} 0 & I \\ \hline -K(t) & -C(t) \end{array}\right] \tag{22}$$

where $C(t)$ and $K(t)$ are $N \times N$ diagonal matrices containing the damping and stiffness coefficients $c(\psi_k)$ and $k(\psi_k)$ of Eq. (20).

The corresponding equation of motion in terms of multiblade coordinates is obtained in the manner of Ref. 5 by using the transformation of Eq. (14) as

$$\dot{y} = Q^{-1}(FQ - \dot{Q})y \stackrel{\Delta}{=} H(t)y \tag{23}$$

The transformation matrix $Q(t)$ of Eq. (19) is not orthogonal, but its inverse is simply obtained by transposing its partitions

$$Q^{-1}(t) = \left[\begin{array}{c|c} M^T(t) & 0 \\ \hline (M(t)P)^T & M^T(t) \end{array}\right] \tag{24}$$

because M is orthogonal and P is skew symmetric.

The coefficient matrix $H(t)$ in Eq. (23) for the multiblade coordinate state can now be expressed explicitly in terms of the damping and stiffness partitions $C(t)$ and $K(t)$ of coefficient matrix $F(t)$ of Eq. (22)

$$H(t) = \left[\begin{array}{c|c} 0 & I \\ \hline -M^TKM - M^TCMP - P^2 & -2P - M^TCM \end{array}\right] \tag{25}$$

The damping and stiffness partitions of $H(t)$ are related to $C(t)$ and $K(t)$ by simple orthogonal transformations and the matrix P. The constant skew-symmetric matrix $-2P$ in the damping partition is a conservative gyroscopic damping term and the constant matrix P^2 in the stiffness partition is a diagonal matrix.

The solutions to the equations of motion are expressed conveniently in terms of the state transition matrices corresponding to the individual blade coordinate state vector and the multiblade coordinate state vector. These state transition matrices can be related using the transformations discussed previously. The individual blade state transition matrix $\Phi(t)$ satisfies Eq. (21)

$$\dot{\Phi}(t) = F(t)\Phi(t); \qquad \Phi(0) = I \tag{26}$$

and the transition matrix $\Psi(t)$ for the multiblade state vector $y(t)$ satisfies Eq. (23)

$$\dot{\Psi}(t) = H(t)\Psi(t); \qquad \Psi(0) = I \tag{27}$$

As shown in Ref. 4 the two transition matrices are related using the present notation by

$$\Psi(t) = Q^{-1}(t)\Phi(t)Q(0) \tag{28}$$

As mentioned in that reference this relationship can be used to prove that the stability of the two systems is the same. In the case that all periodic terms are retained and Floquet theory is applied the Floquet multipliers [the eigenvalues of the transition matrices $\Phi(2\pi)$ and $\Psi(2\pi)$] are the same for the two systems. This is because $Q(t)$ is periodic; $Q(2\pi) = Q(0)$ and $\Psi(2\pi)$ is related to $\Phi(2\pi)$ by a similarity transformation (Eq. 28) under which the eigenvalues are invariant.

The relationship of Eq. (28) along with the simplifications due to the fact that $M(t)$ is orthogonal can be used to express the partitions of the transition matrix $\Psi(t)$ in terms of the partitions of $\Phi(t)$ or vice versa. If the individual blade transition matrix $\Phi(t)$ is represented by the partitions

$$\Phi(t) = \left[\begin{array}{c|c} N(t) & S(t) \\ \hline T(t) & U(t) \end{array}\right] \tag{29}$$

The transition matrix $\Psi(t)$ for the multiblade coordinates is

$$\Psi(t) = \left[\begin{array}{c|c} M^T(NM_0 + SM_0P) & M^TSM_0 \\ \hline -PM^T(NM_0 + SM_0P) + M^T(TM_0 + UM_0P) & (M^TU - PM^TS)M_0 \end{array}\right] \tag{30}$$

where the time argument has been suppressed and $M_0 = M(0)$.

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References

[1]Coleman, R.P., "Theory of Self-Excited Mechanical Oscillations of Helicopter Rotors with Hinged Blades," NACA 1351, 1958.

[2]Hohenemser, K.H. and Yin, S-K., "Some Applications of the Method of Multiblade Coordinates," *Journal of the American Helicopter Society,* Vol. 17, July 1972, pp. 3-12.

[3]Biggers, J.C., "Some Approximations to the Flapping Stability of Helicopter Rotors," *Journal of the American Helicopter Society,* Vol. 14, Oct. 1974, pp 24-33.

[4]Gaonkar, G.H. and Peters, D.A., "Flap-lag Stability with Dynamic Inflow by the Method of Multiblade Coordinates," AIAA/ASME/ASCE/AHS 20th Structures, Structural Dynamics, and Materials Conference, St. Louis, Mo., AIAA Paper 79-0729, April 1979.

[5]Gaonkar, G.H. and Peters, D.A., "Use of Multiblade Coordinates for Helicopter Flap-lag Stability with Dynamic Inflow," *Journal of Aircraft,* Vol. 17, Feb. 1980, pp. 112-118.

[6]Hodges, D.H., "Aeromechanical Stability of Helicopters with a Bearingless Main Rotor-Part I: Equations of Motion," NASA TM 78459, 1978.

[7]Briczinski, S.J. and Cooper, D.E., "Flight Investigation of Rotor/Vehicle State Feedback," NASA CR-132546, 1975.

[8]Gaonkar, G.H., "A General Method with Shaping Filters to Study Random Vibration Statistics of Lifting Rotors with Feedback Controls," *Journal of Sound and Vibration,* Vol. 21, No. 2, 1972, pp. 213-225.

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Aircraft Engine Combustor Casing Life Simulation Evaluation

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Introduction

A COMBUSTOR casing was tested at room temperature under cyclic pressure conditions to investigate: 1) the nature, location, and orientation of life-limiting defects and/or stress concentrations (weld offset mismatch, reinforcement, and the role of bosses); 2) the feasibility of predicting the component life from existing test specimen data; and 3) the stress distribution and the nominal stress range effective for life analysis at the maximum operative pressures.

The need for a component test was also prompted by the fact that fatigue crack growth in the vicinity of welded joints (let alone crossed welds) is complicated, not only by the stress concentrations due to weld geometry effects (mismatch, offset, etc.), but also because the crack may successively enter layers that differ significantly in their crack growth resistance. This clearly leads to a crack-path dependence of the predicted component life. In this case, the total course of a crack must be either known or predictable in order to integrate life increments by fracture mechanics methods. This, in turn, requires rigorous knowledge of the stress and defect distributions in critical areas. In the lack of such a detailed knowledge of stress distributions and defect populations, life analysis often resorts reasonably to the weakest link concept, which assumes that the crack will grow along, or ultimately "seek," the weakest plane in the weld.

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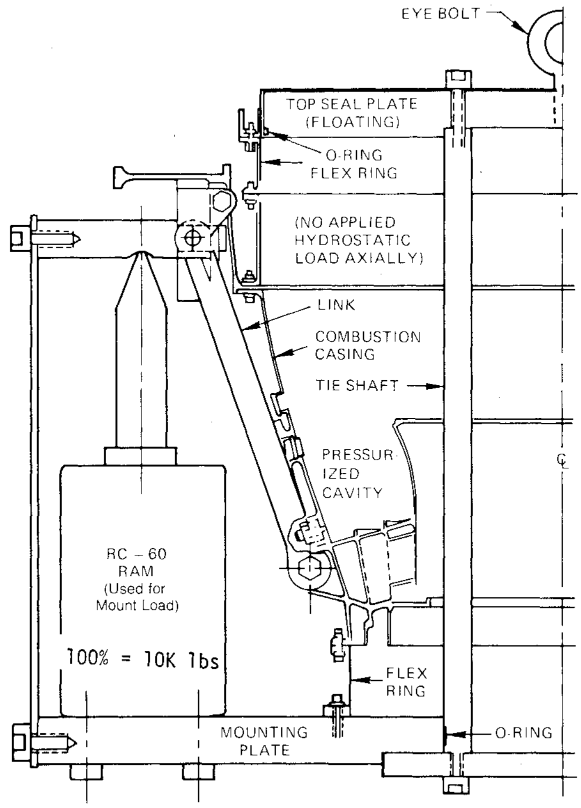

Fig. 1 Diagram of test arrangement.

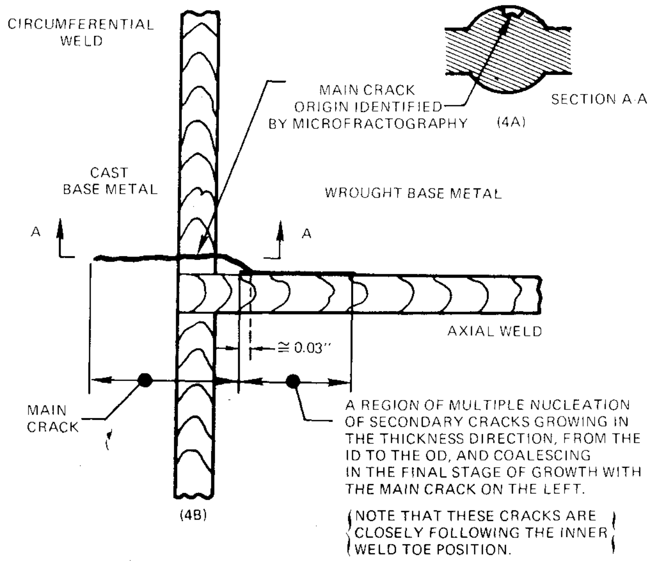

Fig. 2 A schematic illustration of the crack path in the combustor casing test (i.d. face in plane).

Table 1 Crack growth history

Cycles	Crack length, in. Fwd	Aft	Total
250			0.190
1500	0.058	0.062	0.280
2000	0.070	0.085	0.315
3000	0.115	0.125	0.400
4000	0.150	0.160	0.460
4500	0.155	0.200	0.515
5000	0.225	0.400	0.785
5500	0.250	0.425	0.835
6000	0.310	0.500	0.968
6200	0.380	0.700	1.24
6500	0.500	0.850	1.51
6600	0.510	0.925	1.6